Symbolic Solving of
Extended Regular Expression Inequalities

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### Definition

$r, s, t := \epsilon \mid A \mid r + s \mid r \cdot s \mid r^* \mid r\&s \mid !r$

- $\Sigma$ is a potentially infinite set of symbols
- $A, B, C \subseteq \Sigma$ range over sets of symbols
- $[r] \subseteq \Sigma^*$ is the language of a regular expression $r$, where $[A] = A$
Language Inclusion

Definition

Given two regular expressions $r$ and $s$,

$$r \subseteq s \iff \overline{[r]} \subseteq \overline{[s]}$$

- $[r] \subseteq [s]$ iff $[r] \cap \overline{[s]} = \emptyset$
- Decidable using standard techniques:
  Construct DFA for $r \&!s$ and check for emptiness
- Drawback is the expensive construction of the automaton
- PSPACE-complete
Deciding containment for *basic regular expressions*

Based on derivatives and expression rewriting

Avoid the construction of an automaton

\[ \partial_a(r) \] computes a regular expression for \( a^{-1}[r] \) (Brzozowski)

with \( u \in [r] \) iff \( \epsilon \in [\partial_u(r)] \)

**Lemma**

For regular expressions \( r \) and \( s \),

\[ r \sqsubseteq s \iff (\forall u \in \Sigma^*) \; \partial_u(r) \sqsubseteq \partial_u(s). \]
Antimirov’s Algorithm (cont’d)

Lemma

\[ r \sqsubseteq s \iff (\nu(r) \Rightarrow \nu(s)) \land (\forall a \in \Sigma) \partial_a(r) \sqsubseteq \partial_a(s) \]

**CC-DISPROVE**

\[
\frac{\nu(r) \land \neg \nu(s)}{r \sqsubseteq s \vdash_{\text{CC}} \text{false}}
\]

**CC-UNFOLD**

\[
\frac{\nu(r) \Rightarrow \nu(s)}{r \sqsubseteq s \vdash_{\text{CC}} \{ \partial_a(r) \sqsubseteq \partial_a(s) \mid a \in \Sigma \}}
\]

- Choice of next step’s inequality is nondeterministic
- An infinite alphabet requires to compute for infinitely many \( a \in \Sigma \)
First Symbols

Lemma

\[ r \sqsubseteq s \iff (\nu(r) \Rightarrow \nu(s)) \land (\forall a \in \text{first}(r)) \partial_a(r) \sqsubseteq \partial_a(s) \]

- Let \( \text{first}(r) := \{ a \mid aw \in [r] \} \) be the set of first symbols
- Restrict symbols to first symbols of the left hand side
- \texttt{CC-Unfold} does not have to consider the entire alphabet
- For extended regular expressions, \( \text{first}(r) \) may still be an infinite set of symbols
Problems

- Antimirov’s algorithm only works with basic regular expressions or requires a finite alphabet.
- Extension of *partial derivatives* (Caron et al.) that computes an NFA from an extended regular expression.
- Works on sets of sets of expressions.
- Computing derivatives becomes more expensive.
Goal

- Algorithm for deciding $[r] \subseteq [s]$ quickly
- Handle *extended regular expressions*
- Deal effectively with very large (or infinite) alphabets (e.g. Unicode character set)

Solution

- Require finitely many atoms, even if the alphabet is infinite
- Compute derivatives with respect to literals
A literal is a set of symbols $A \subseteq \Sigma$

**Definition**

$A$ is an element of an *effective* boolean algebra $(U, \sqcup, \sqcap, \bar{\cdot}, \bot, \top)$ where $U \subseteq \wp(\Sigma)$ is closed under the boolean operations.

- For finite (small) alphabets:
  
  $U = \wp(\Sigma)$, $A \subseteq \Sigma$

- For infinite (or just too large) alphabets:
  
  $U = \{ A \in \wp(\Sigma) \mid A \text{ finite} \lor \bar{A} \text{ finite}\}$

- Second-level regular expressions:
  
  $\Sigma \subseteq \wp(\Gamma^*)$ with $U = \{ A \subseteq \wp(\Gamma^*) \mid A \text{ is regular}\}$

- Formulas drawn from a first-order theory over alphabets
  
  For example, $[a-z]$ represented by $x \geq 'a' \land x \leq 'z'$
Derivatives with respect to Literals

- Definition for $\partial_A(r)$?
- $\partial_a(r)$ computes a regular expression for $a^{-1}[r]$ (Brzozowski)

Desired property

\[
[\partial_A(r)] = A^{-1}[r] = \bigcup_{a \in A} a^{-1}[r] = \bigcup_{a \in A} [\partial_a(r)]
\]
Positive Derivatives on Literals

Definition

\[ \delta_A^+(B) := \begin{cases} \epsilon, & B \cap A \neq \perp \\ \emptyset, & \text{otherwise} \end{cases} \]

Problem

With \( A = \{a, b\} \) and \( r = (a \cdot c) \& (b \cdot c) \),

\[ \delta_A^+(r) = \delta_A^+(a \cdot c) \& \delta_A^+(b \cdot c) = c \& c \subseteq \emptyset \]
Negative Derivatives on Literals

Definition

\[ \delta_A^{-}(B) := \begin{cases} \epsilon, & \overline{B} \cap A = \bot \\ \emptyset, & \text{otherwise} \end{cases} \]

Problem

With \( A = \{a, b\} \) and \( r = (a \cdot c) + (b \cdot c) \),

\[ \delta_A^{-}(r) = \delta_A^{-}(a \cdot c) + \delta_A^{-}(b \cdot c) = \emptyset + \emptyset \subseteq c \]
Positive and Negative Derivatives

- Extends Brzozowski’s derivative operator to sets of symbols.
- Defined by induction and flip on the complement operator

**Definition**

From $\partial_a(!s) = !\partial_a(s)$, define:

$$
\delta^+_A(!r) := !\delta^+_A(r) \quad \text{and} \quad \delta^-_A(!r) := !\delta^-_A(r)
$$

**Lemma**

*For any regular expression $r$ and literal $A$,*

$$
[\delta^+_A(r)] \supseteq \bigcup_{a \in A} [\partial_a(r)] \quad \text{and} \quad [\delta^-_A(r)] \subseteq \bigcap_{a \in A} [\partial_a(r)]
$$
Literals of an Inequality

Lemma

\[ r \sqsubseteq s \iff (\nu(r) \Rightarrow \nu(s)) \land (\forall a \in \text{first}(r)) \partial_a(r) \sqsubseteq \partial_a(s) \]

- first\((r)\) may still be an infinite set of symbols
- Use first literals as representatives of the first symbols

Example

1. Let \( r = \{a, b, c, d\} \cdot d^* \), then \( \{a, b, c, d\} \) is a first literal
2. Let \( s = \{a, b, c\} \cdot c^* + \{b, c, d\} \cdot d^* \), then \( \{a, b, c\} \) and \( \{b, c, d\} \) are first literals
Problem

Let \( r = \{a, b, c, d\} \cdot d^* \), \( s = \{a, b, c\} \cdot c^* + \{b, c, d\} \cdot d^* \), and \( A = \{a, b, c, d\} \), then

\[
\delta_A^+(r) \subseteq \delta_A^+(s) \quad (1)
\]
\[
\delta_A^+((a, b, c, d) \cdot d^*) \subseteq \delta_A^+((a, b, c) \cdot c^*) + \delta_A^+((b, c, d) \cdot d^*) \quad (2)
\]
\[
d^* \subseteq c^* + d^* \quad (3)
\]

- Positive (negative) derivatives yield an upper (lower) approximation
- To obtain the precise information, we need to restrict these literals suitably to next literals, e.g. \( \{\{a\}, \{b, c\}, \{d\}\} \)
Next Literals

\begin{align*}
\text{next}(\epsilon) &= \{\emptyset\} \\
\text{next}(A) &= \{A\} \\
\text{next}(r + s) &= \text{next}(r) \times \text{next}(s) \\
\text{next}(r \cdot s) &= \begin{cases} \\
\text{next}(r) \times \text{next}(s), & \nu(r) \\
\text{next}(r), & \neg \nu(r) \\
\end{cases} \\
\text{next}(r^*) &= \text{next}(r) \\
\text{next}(r \& s) &= \text{next}(r) \cap \text{next}(s) \\
\text{next}(!r) &= \text{next}(r) \cup \{\bigcap \{\overline{A} \mid A \in \text{next}(r)\}\}
\end{align*}

\textbf{Definition}

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two sets of disjoint literals.

\[\mathcal{L}_1 \times \mathcal{L}_2 := \{(A_1 \cap A_2), (A_1 \cap \bigcap \mathcal{L}_2), (\bigcap \mathcal{L}_1 \cap A_2) \mid A_1 \in \mathcal{L}_1, A_2 \in \mathcal{L}_2\}\]
Example

Let \( s = \{a, b, c\} \cdot c^* + \{b, c, d\} \cdot d^* \), then

\[
\text{next}(s) = \text{next}(\{a, b, c\} \cdot c^*) \bowtie \text{next}(\{b, c, d\} \cdot d^*) \\
= \{\{a, b, c\} \bowtie \{b, c, d\}\} \\
= \{\{a\}, \{b, c\}, \{d\}\}
\]

Lemma

For all \( r \),

- \( \bigcup \text{next}(r) \supseteq \text{first}(r) \)
- \( |\text{next}(r)| \) is finite
- \((\forall A, B \in \text{next}(r)) \ A \cap B = \emptyset\)
Lemma

Let $\mathcal{L} = \text{next}(r)$ and $A \in \text{next}(r) \setminus \{\emptyset\}$.

1. $(\forall a, b \in A) \partial_a(r) = \partial_b(r) \land \delta_A^+(r) = \delta_A^-(r) = \partial_a(r)$

2. $(\forall a \notin \bigcup \mathcal{L}) \partial_a(r) = \emptyset$

Definition

Let $A' \in \text{next}(r)$. For each $\emptyset \neq A \subseteq A'$ define $\partial_A(r) := \partial_a(r)$, where $a \in A$. 
Next Literals of an Inequality

- *Next literal* of $\text{next}(r \sqsubseteq s)$
- Sound to join literals of both sides $\text{next}(r) \Join \text{next}(s)$
- Contains also symbols from $s$
- First symbols of $r$ are sufficient to prove containment

**Definition**

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two sets of disjoint literals.

$$\mathcal{L}_1 \Join \mathcal{L}_2 := \{(A_1 \cap A_2), (A_1 \cap \bigcup \mathcal{L}_2) \mid A_1 \in \mathcal{L}_1, A_2 \in \mathcal{L}_2\}$$

Left-based join corresponds to $\text{next}(r \& (!s))$.

**Definition**

Let $r \sqsubseteq s$ be an inequality, define: $\text{next}(r \sqsubseteq s) := \text{next}(r) \Join \text{next}(s)$
Solving Inequalities

Lemma

\[ r \subseteq s \iff (\nu(r) \Rightarrow \nu(s)) \land (\forall a \in \text{first}(r)) \partial_a(r) \subseteq \partial_a(s) \]

To determine a finite set of representatives
- select one symbol \( a \) from each equivalence class \( A \in \text{next}(r) \)
- calculate with \( \delta_+^A(r) \) or \( \delta_-^A(r) \) with \( A \in \text{next}(r) \)

Theorem (Containment)

\[ r \subseteq s \iff (\nu(r) \Rightarrow \nu(s)) \land (\forall A \in \text{next}(r \sqsubseteq s)) \partial_A(r) \subseteq \partial_A(s) \]
Conclusion

- Generalize Brzozowski’s derivative operator
- Extend Antimirov’s algorithm for proving containment
- Provides a symbolic decision procedure that works with extended regular expressions on infinite alphabets
- Literals drawn from an effective boolean algebra
- Main contribution is to identify a finite set that covers all possibilities
Regular Languages

The language \([r] \subseteq \Sigma^*\) of a regular expression \(r\) is defined inductively by:

\[
\begin{align*}
[\epsilon] & = \{\epsilon\} \\
[A] & = \{a \mid a \in A\} \\
[r+s] & = [r] \cup [s] \\
[r \cdot s] & = [r] \cdot [s] \\
[r^*] & = [r] \cdot [r^*] \\
[r \& s] & = [r] \cap [s] \\
[!r] & = [r]
\end{align*}
\]
Nullable

The *nullable* predicate $\nu(r)$ indicates whether $\llbracket r \rrbracket$ contains the empty word, that is, $\nu(r)$ iff $\epsilon \in \llbracket r \rrbracket$.

\[
\begin{align*}
\nu(\epsilon) & = \text{true} \\
\nu(A) & = \text{false} \\
\nu(r+s) & = \nu(r) \lor \nu(s) \\
\nu(r \cdot s) & = \nu(r) \land \nu(s) \\
\nu(r^*) & = \text{true} \\
\nu(r\&s) & = \nu(r) \land \nu(s) \\
\nu(!r) & = \neg \nu(r)
\end{align*}
\]
Brzozowski Derivatives

\[ \partial_a(r) \] computes a regular expression for the left quotient \( a^{-1}[r] \).

\[
\begin{align*}
\partial_a(\epsilon) &= \emptyset \\
\partial_a(A) &= \begin{cases} 
\epsilon, & a \in A \\
\emptyset, & a \notin A 
\end{cases} \\
\partial_a(r+s) &= \partial_a(r) + \partial_a(s) \\
\partial_a(r \cdot s) &= \begin{cases} 
\partial_a(r) \cdot s + \partial_a(s), & \nu(r) \\
\partial_a(r) \cdot s, & \neg \nu(r)
\end{cases} \\
\partial_a(r^*) &= \partial_a(r) \cdot r^* \\
\partial_a(r \& s) &= \partial_a(r) \& \partial_a(s) \\
\partial_a(\neg r) &= \neg \partial_a(r)
\end{align*}
\]
Let $\text{first}(r) := \{ a \mid aw \in [r] \}$ be the set of first symbols derivable from regular expression $r$.

\[
\begin{align*}
\text{first}(\epsilon) &= \emptyset \\
\text{first}(A) &= A \\
\text{first}(r + s) &= \text{first}(r) \cup \text{first}(s) \\
\text{first}(r \cdot s) &= \begin{cases} \\
\text{first}(r) \cup \text{first}(s), & \nu(r) \\
\text{first}(r), & \neg \nu(r) \\
\end{cases} \\
\text{first}(r^*) &= \text{first}(r) \\
\text{first}(r \& s) &= \text{first}(r) \cap \text{first}(s) \\
\text{first}(!r) &= \Sigma \setminus \{ a \in \text{first}(r) \mid \partial_a(r) \neq \Sigma^* \}
\end{align*}
\]
Let first($r$) := \{a | aw \in [r]\} be the set of first symbols derivable from regular expression $r$.

\[
\begin{align*}
literal(\epsilon) &= \emptyset \\
literal(A) &= \{A\} \\
literal(r+s) &= \literal(r) \cup \literal(s) \\
literal(r \cdot s) &= \begin{cases} \\
\literal(r) \cup \literal(s), & \nu(r) \\
\literal(r), & \neg \nu(r)
\end{cases} \\
literal(r^*) &= \literal(r) \\
literal(r \& s) &= \literal(r) \cap \literal(s) \\
literal(!r) &= \Sigma \cap \bigcup \{A \in \literal(r) \mid \partial_A(r) = \Sigma^*\}
\end{align*}
\]
Lemma (Coverage)

For all $a$, $u$, and $r$ it holds that:

$$ u \in \delta_a(r) \iff \exists A \in \text{next}(r) : a \in A \land u \in \delta_A^+(r) \land u \in \delta_A^-(r) $$
Theorem (Finiteness)

Let $R$ be a finite set of regular inequalities. Define

$$F(R) = R \cup \{ \partial_A(r \sqsubseteq s) \mid r \sqsubseteq s \in R, A \in \text{next}(r \sqsubseteq s) \}$$

For each $r$ and $s$, the set $\bigcup_{i \in \mathbb{N}} F^{(i)}(\{ r \sqsubseteq s \})$ is finite.
Decision Procedure for Containment

\( (\text{Disprove}) \)

\[
\begin{align*}
\nu(r) & \quad \neg\nu(s) \\
\Gamma \vdash r \sqsubseteq s : \text{false}
\end{align*}
\]

\( (\text{Cycle}) \)

\[
\begin{align*}
r \sqsubseteq s & \in \Gamma \\
\Gamma \vdash r \sqsubseteq s : \text{true}
\end{align*}
\]

\( (\text{Unfold-True}) \)

\[
\begin{align*}
r \sqsubseteq s & \not\in \Gamma \quad \nu(r) \Rightarrow \nu(s) \\
\forall A \in \text{next}(r \sqsubseteq s) : \Gamma \cup \{r \sqsubseteq s\} & \vdash \partial_A(r) \sqsubseteq \partial_A(s) : \text{true} \\
\Gamma & \vdash r \sqsubseteq s : \text{true}
\end{align*}
\]

\( (\text{Unfold-False}) \)

\[
\begin{align*}
r \sqsubseteq s & \not\in \Gamma \quad \nu(r) \Rightarrow \nu(s) \\
\exists A \in \text{next}(r \sqsubseteq s) : \Gamma \cup \{r \sqsubseteq s\} & \vdash \partial_A(r) \sqsubseteq \partial_A(s) : \text{false} \\
\Gamma & \vdash r \sqsubseteq s : \text{false}
\end{align*}
\]
(Prove-Identity) \[ \Gamma \vdash r \sqsubseteq r : true \]

(Prove-Nullable) \[ \nu(s) \quad \quad \quad \]
\[ \Gamma \vdash \epsilon \sqsubseteq s : true \]

(Prove-Empty) \[ \Gamma \vdash \emptyset \sqsubseteq s : true \]

(Disprove-Empty) \[ \exists A \in \text{next}(r) : A \neq \emptyset \]
\[ \Gamma \nvdash r \sqsubseteq \emptyset : false \]
**Soundness**

### Theorem (Soundness)

*For all regular expression $r$ and $s$:

$$
\emptyset \vdash r \sqsubseteq s : \top \iff r \sqsubseteq s
$$

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*Regular Expression Inequalities*

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Let \( r = \{a, b, c, d\} \cdot d^* \), \( s = \{a, b, c\} \cdot d^* + \{b, c, d\} \cdot d^* \), and \( A = \{a, b, c, d\} \), then

\[
\delta_A^-(r) \subseteq \delta_A^+(s) 
\]

\[
\delta_A^-(\{a, b, c, d\} \cdot d^*) \subseteq \delta_A^-(\{a, b, c\} \cdot d^*) + \delta_A^-(\{b, c, d\} \cdot d^*) 
\]

\[
d^* \subseteq \emptyset + \emptyset 
\]
Next Literals of an Inequality

Example

Let \( r = \{a, b, c, d\} \cdot d^* \), \( s = \{a, b, c\} \cdot c^* + \{b, c, d\} \cdot d^* \) then

\[
\text{next}(r \sqsubseteq s) = \text{next}(\{a, b, c, d\} \cdot d^*) \ltimes \text{next}(\{a, b, c\} \cdot d^* + \{b, c, d\} \cdot d^*)
= \{\{a\}, \{b, c\}, \{d\}\}
\]
Incomplete Containment

Conjecture

\[ r \sqsubseteq s \iff (\nu(r) \Rightarrow \nu(s)) \land (\forall A \in \text{literal}(r)) \delta^+_A(r) \sqsubseteq \delta^-_A(s) \]